

Volterra Equations of the First kind with Discontinuous Kernels in the Theory of Evolving Systems Control

D. N. Sidorov*

** Melentiev Energy Systems Institute
Russian Academy of Sciences
130 Lermontov Street, 664033 Irkutsk
Irkutsk State University
20 Boulevard Gagarin, 664003 Irkutsk
dsidorov@isem.sei.irk.ru*

Abstract. *The Volterra integral equations of the first kind with piecewise smooth kernel are considered. Such equations appear in the theory of optimal control of the evolving systems. The existence theorems are proved. The method for constructing approximations of parametric families of solutions of such equations is suggested. The parametric family of solutions is constructed in terms of a logarithmic power asymptotics.*

Keywords: *Volterra equations, discontinuous kernel, successive approximations, optimal lifetime, evolving systems*

Introduction

The theory of integral models of evolving systems was first initiated by V. Glushkov in the 70s of XX century. Readers may refer to the bibliography in [HY96, Apa03, DL95, MST11, HY03]. Such theory employs the Volterra integral equations of the first kind where limit of integration is time function. The theory and numerical methods of such nonclassic equations were studied in the monograph [Apa03] and

applied for estimation and optimization of lifetime of electric power systems (EPS) components [MST11].

In this paper we address the following integral equation:

$$\int_0^t K(t, s)x(s)ds = f(t), \quad 0 < t \leq T, \quad (1)$$

where kernel is defined as follows:

$$K(t, s) = \begin{cases} K_1(t, s), & 0 \leq s < \alpha_1(t), \\ K_2(t, s), & \alpha_1(t) \leq s < \alpha_2(t), \\ \dots & \dots\dots\dots \\ K_n(t, s), & \alpha_{n-1}(t) \leq s \leq t, \end{cases} \quad (2)$$

$$0 < \alpha_1(t) < \alpha_2(t) < \dots < \alpha_{n-1}(t) < t, \quad |\alpha'_i(t)| < 1.$$

Functions $K_i(t, s), i = \overline{1, n}, f(t)$ are continuous and sufficiently smooth, $f(0) = 0$.

The Glushkov integral model of evolving systems [HY96, MST11, HY03] is the special case of the Volterra integral equation (1) – (2) when all the kernels except $K_1(t, s)$ are zeros.

First results in studies of the Volterra equations with discontinuous kernels were formulated by G.C. Evans [Eva10] in the beginning of XX century. Results in the spectral theory of integral operators with discontinuous kernels were obtained by A.P. Khromov in [Khr06]. Asymptotic approximations of solutions of the Volterra equations of the first kind with analytical kernel $K(t, s)$ were constructed by N.A. Magnitsky [Mag83].

It is to be noted that solutions of the equations (1) can have an arbitrary constants and can be unbounded for $t \rightarrow 0$. For example, if

$$K(t, s) = \begin{cases} 1, & 0 \leq s < t/2, \\ -1, & t/2 \leq s \leq t, \end{cases} \quad (3)$$

$f(t) = t$, then equation (1) has the solution $x(t) = c - \frac{\ln t}{\ln 2}$, where c is constant.

In this paper we employ results of the papers [Mag83, SS06, ST09, SST07] in order to formulate the algorithm for construction of the continuous solutions of equation (1) for $0 < t \leq T$ in the following form:

$$x(t) = \sum_{i=0}^N x_i(\ln t) t^i + t^N u(t). \quad (4)$$

Coefficients $x_i(\ln t)$ are constructed as polynomials on powers of $\ln t$ and they may depend on certain number of arbitrary constants. N defines the necessary smoothness of the functions $K_i(t, s)$, $f(t)$. In this paper we propose an algorithm for construction of the function $u(t)$ in representation of the desired solution (4) based on successive approximations method which is uniformly converge on $[0, T]$. It is to be noted, that logarithmic-power asymptotics have been efficiently employed for solution of integral and differential equations in irregular cases [ST09] – [SS11], [Mag83].

The paper is organized as follows. In Section 1 after the problem statement we introduce the structure of solution and prove the existence theorem. The method for the asymptotic approximations construction is suggested in Section 2. The main theorem is formulated and proved also in Section 2. Finally, concluding remarks are given.

1. The structure of solutions and existence theorem

For sake of clarity let us suppose that $\alpha_i(t) = \alpha_i t$, where $0 = \alpha_0 < \alpha_1 < \dots < \alpha_n = 1$. Let us introduce the condition

A. $K_n(t, t) \neq 0$ for $t \in [0, T]$ and N is selected to fulfill the following inequality

$$\max_{0 \leq t \leq T} |K_n(t, t)|^{-1} \sum_{i=1}^n \left(\alpha_i^{1+N} |K_i(t, \alpha_i t)| + \right. \quad (5)$$

$$\left. + \alpha_{i-1}^{1+N} |K_i(t, \alpha_{i-1} t)| \right) \leq 1 + q,$$

where $q < 1$, $\alpha_0 = 0$, $\alpha_n = 1$. Condition **A** is fulfilled for large enough N since $\alpha_i \in (0, 1)$ for $i = \overline{1, n-1}$

Lemma 1.1 *Let condition **A** be fulfilled, let all the functions $K_i(t, s), i = \overline{1, n}$ be differentiable wrt t and continuous wrt s . Then the homogenous equation*

$$\int_0^t K(t, s) s^N u(s) ds = 0 \quad (6)$$

has the trivial solution in the space $\mathbb{C}_{[0, T]}$.

Proof 1.1 *Let us differentiate the equation (6) and take into account (2). Then we get an equivalent integral-functional equation*

$$Lu + \sum_{i=1}^n \int_{\alpha_{i-1}t}^{\alpha_i t} \frac{K'_i(t, s)}{K_n(t, t)} \left(\frac{s}{t}\right)^N u(s) ds = 0, \quad (7)$$

where

$$Lu = \sum_{i=1}^n \left(\alpha_i^{1+N} K_i(t, \alpha_i t) u(\alpha_i t) - \alpha_{i-1}^{1+N} K_i(t, \alpha_{i-1} t) u(\alpha_{i-1} t) \right) (K_n(t, t))^{-1}.$$

Due to the condition **A** in the space $\mathbb{C}_{[0, T]}$ we have the following estimate

$$\|Lu - u\| \leq q \|u\|.$$

Therefore, according to the theorem on the inverse operator ([Tre07], p.134), and because of the inequalities $0 = \alpha_0 < \alpha_1 < \dots < \alpha_n = 1$ there exists the following bounded inverse operator $L^{-1} \in \mathcal{L}(\mathbb{C}_{[0, T]} \rightarrow \mathbb{C}_{[0, T]})$

$$\|L^{-1}\| \leq \frac{1}{1-q} \quad (8)$$

and equation (6) can be reduced as follows

$$u(t) = -L^{-1} \sum_{i=1}^n \int_{\alpha_{i-1}t}^{\alpha_i t} \frac{K'_i(t, s)}{K_n(t, t)} (s/t)^N u(s) ds \equiv Au, \quad (9)$$

where $0 \leq t \leq T$. Let us introduce the following equivalent norm $\|u\| = \max_{0 \leq t \leq T} e^{-lt} |u(t)|$, $l > 0$ in the space $\mathbb{C}_{[0,T]}$. In this norm the inequality (8) remains correct and for sufficiently large l an operator A will be contracting since $\|A\| \leq \frac{1}{1-q} m(l)$, where $m(l) \rightarrow 0$ for $l \rightarrow +\infty$. Therefore homogeneous equation (9) has trivial solution.

Corollary 1.1 Let all the condition of Lemma 1 be fulfilled, $g(t) \in \mathbb{C}_{[0,T]}^{(1)}$, $|g'(t)| = o(t^N)$ for $t \rightarrow +0$. Then inhomogeneous equation $\int_0^t K(t,s) s^N u(s) ds = g(t)$ has the unique solution, and $u(t) \rightarrow 0$ for $t \rightarrow +0$.

Proof is trivial since differentiation of this equation leads to the equivalent equation

$$u(t) = Au + t^{-N} L^{-1} g'(t) \quad (10)$$

with contracting operator A and continuous free function.

Theorem 1.1 Let in the space of continuous on $(0, T]$ functions which have the finite limit for $t \rightarrow +0$ (briefly, in class $\mathbb{C}_{(0,T]}$) exists the function $x^N(t)$ such as for $t \rightarrow +0$

$$\left(- \int_0^t K(t,s) x^N(s) ds + f(t) \right)' = o(t^N),$$

$f(t) \in \mathbb{C}_{[0,T]}^{(1)}$, $f(0) = 0$. Then equation (1) has the following solution

$$x(t) = x^N(t) + t^N u(t) \quad (11)$$

in class $\mathbb{C}_{(0,T]}$. Here function $u(t) \in \mathbb{C}_{[0,T]}$, $u(t) \rightarrow 0$ for $t \rightarrow +0$ and it can be uniquely constructed with successive approximations method.

Proof 1.2 Proof follows from the corollary 1. Indeed, with (11) we can rewrite the equation (1) as follows

$$\int_0^t K(t,s) s^N u(s) ds = g(t), \quad (12)$$

where function $g(t)$ is following

$$g(t) = - \int_0^t K(t, s)x^N(s)ds + f(t) \quad (13)$$

and it satisfies the condition of the corollary 1. Therefore in (11) the function $u(t)$ can be uniquely constructed with successive approximations from the equation (10) using arbitrary initial condition.

Definition 1.1 The equation (12) with right hand side (13) we call regularization of the equation (1), and function $x^N(t)$ as asymptotic approximation of solution (11) of the equation (1).

It is to be noted that one could numerically find the function $u(t)$ by solution of the equation (12) based on well known numerical quadrature schemes (see e.g. the bibliography in the monograph [Apa03]). The method of constructing asymptotic approximations $x^N(t)$ in the solution (12) we will study below in the Section 2.

2. The method of asymptotic approximations construction

Let us suppose that along with the condition **A** the condition **B** be fulfilled. Functions $K_i(t, s), i = \overline{1, n}, f(t)$ are $N + 1$ times differentiable in the neighborhood of zero, where N is selected according to the condition **A**. We introduce an auxiliary algebraic equation wrt $j \in \mathbb{N}$

$$L(j) \triangleq \sum_{i=1}^n K_i(0, 0)(\alpha_i^{1+j} - \alpha_{i-1}^{1+j}) = 0 \quad (14)$$

and name it as *characteristic equation* of the integral equation (1). Since $f(0) = 0$, then equation

$$\begin{aligned} & \sum_{i=1}^n (\alpha_i K_i(t, \alpha_i t)x(\alpha_i t) - \alpha_{i-1} K_i(t, \alpha_{i-1} t)x(\alpha_{i-1} t)) + \\ & + \sum_{i=1}^n \int_{\alpha_{i-1} t}^{\alpha_i t} K_i'(t, s)x(s)ds = f'(t) \end{aligned}$$

is equivalent to the equation (1). We will look for the asymptotical approximation of it's solution as following polynomial $x^N(t) = \sum_{j=0}^N x_j(\ln t)t^j$. Based on the method of undetermined coefficients, and taking into account the inequalities $0 = \alpha_0 < \alpha_1 < \dots < \alpha_n = 1$ we construct the recursive sequence of difference equations wrt the coefficients $x_j(z)(z = \ln t)$ as follows:

$$K_n(0, 0)x_j(z) + \sum_{i=1}^{n-1} \alpha_i^{1+j} (K_i(0, 0) - K_{i+1}(0, 0))x_j(z + a_i) = M_j(x_0, \dots, x_{j-1}), \quad (15)$$

where $j = \overline{0, N}$, $a_i = \ln \alpha_i$, $i = \overline{1, n-1}$, $M_0 = f'(0)$.

Here we follow ([Gel59], p.330) and we seek the solution of the homogeneous difference equations in the form of $x = \lambda^z$. Substitution of the function λ^z into the homogenius difference equations leads to $N+1$ equations for difference equations (15):

$$\mathcal{P}_j(\lambda) \equiv K_n(0, 0) + \sum_{i=1}^{n-1} \alpha_i^{1+j} (K_i(0, 0) - K_{i+1}(0, 0))\lambda^{a_i} = 0, \quad j = \overline{0, N}. \quad (16)$$

Therefore we have

Property 1. *j th equation (16) has the root $\lambda = 1$ if and only if j satisfies characteristic equation (14) of the integral equation (1). Moreover, multiplicity of the root j of the equation (14) is equal to r_j iff*

$$L(j) = \sum_{i=1}^n K_i(0, 0)(\alpha_i^{1+j} - \alpha_{i-1}^{1+j}) = 0, \quad (17)$$

$$\sum_{i=1}^{n-1} \alpha_i^{1+j} (K_i(0, 0) - K_{i+1}(0, 0))a_i^l = 0, \quad l = \overline{1, r_j - 1}, \quad (18)$$

$$\sum_{i=1}^{n-1} \alpha_i^{1+j} (K_i(0, 0) - K_{i+1}(0, 0))a_i^{r_j} \neq 0,$$

where $\alpha_0 = 0, \alpha_n = 1, a_0 = 0, a_n = 0, a_i = \ln \alpha_i, i = \overline{1, n-1}$, and multiplicity $r_j \leq n-1$.

Proof follows from the equality

$$\sum_{i=1}^{n-1} \alpha_i^{1+j} K_{i+1}(0, 0) = \sum_{i=2}^n \alpha_{i-1}^{1+j} K_i(0, 0) \quad (19)$$

and from the structure of the equations (14), (16).

If we suppose that for certain j multiplicity $r_j \geq n$, then $K_1(0, 0) = K_2(0, 0) = \dots = K_{n-1}(0, 0) = K_n(0, 0)$ due to (18), since $\det \|a_i^l\|_{i,l=\overline{1,n}} \neq 0$. But due to (17) $K_n(0, 0) = 0$, which contradicts **A**.

Under the conditions **A**, **B** there are two cases.

2.1. Regular case

Let $L(j) \neq 0, j \in \mathbb{N}$. Then $\lambda = 1$ does not satisfy any of the equations in the sequence (16). All the coefficients x_i of the asymptotics $x^N = \sum_{i=0}^N x_i t^i$ can be determined uniquely with method of undetermined coefficients and do not depend upon $\ln t$.

Therefore we have the following theorem

Theorem 2.1 *Let the conditions **A**, **B** and $L(j) \neq 0, j \in \mathbb{N}$ be fulfilled.*

Then equation (1) has in $\mathbb{C}_{[0,T]}$ the solution $x(t) = \sum_{i=0}^N x_i t^i + t^N u(t)$, where x_i are determined uniquely with method of undetermined coefficients and function $u(t)$ is uniquely constructed (numerically or with successive approximations from equation (12)).

2.2. Irregular case

Let $L(j) = 0$ only for $j \in \{j_1, \dots, j_k\} \subset \{0, 1, \dots, N\}$ and multiplicity of the root $\lambda = 1$ for the corresponding characteristic equation is r_j . Let in j th difference equation (15) right hand side $M_j(z)$ appears to

be polynomial from z of the order $n_j \geq 0$. Then in irregular case, i.e. for $r_j \geq 1$, based on ([Gel59], p.338) particular solution of the j th equation (15) we have to search as following polynomial $\hat{x}(z) = \sum_{i=r_j}^{n_j+r_j} c_i z^i$.

Coefficients c_i of this polynomial can be sequentially calculated by the method of undetermined coefficients starting from $c_{n_j+r_j}$. Coefficient $x_j(z)$ of desired asymptotical approximation x^N in this case is as follows

$$x_j(z) = c_0 + c_1 z + \dots + c_{r_j-1} z^{r_j-1} + \hat{x}(z).$$

In irregular case when $r_j \geq 1$, constants c_0, \dots, c_{r_j-1} remain arbitrary since functions z^i , $i = 0, 1, \dots, r_j-1$ satisfy j th homogenous difference equation corresponding to (15).

In applications one could use the Propetry 1, and coefficient $x_j(z)$ in irregular case directly as polynomial $\sum_{i=0}^{n_j+r_j} c_i z^i$, where $c_{n_j+r_j}, \dots, c_0$ are determined sequentially using method of undetermined coefficients. And c_{r_j-1}, \dots, c_0 remains arbitrary. Therefore in irregular case when $L(j) = 0$ for some j new arbitrary constants r_j appear in determination of the coefficient $x_j(z)$. Order of the polynomial $x_j(z)$ on the value of the multiplicity of r_j or root $\lambda = 1$ of the j th equation (16) becomes greater than order n_j of the right hand side of the corresponding equation (15), i.e. of the order of the polynomial $M_j(z)$.

Therefore we have the following theorem:

Theorem 2.2 *Let the conditions **A**, **B** are fulfilled. Let characteristic equation $L(j) = 0$ of integral equation (1) has exactly k natural roots $\{j_1, \dots, j_k\}$. And let the root $\lambda = 1$ of j th equation (16) has multiplicity r_j . Then equation (1) has the following solution in $\mathbb{C}_{(0,T]}$*

$$x = \sum_{i=0}^N x_i(\ln t) t^i + t^N u(t), \quad (20)$$

which depends on $p = r_1 + \dots + r_k$ arbitrary constants. Moreover, coefficients x_i of the asymptotic approximation $x^N(t)$ are polynomials from $\ln t$. Function $u(t)$ can be constructed by successive approximations which converge uniformly for $t \in [0, T]$, or numerically from (12).

Remark 2.1 If $L(0) = 0$, then in solution (20) $x_0 = \text{const} + a \ln t$, where a is the defined constant. Therefore in this case $x(t) \in \mathbb{C}_{(0,T]}$, $\lim_{t \rightarrow +0} x(t) = \infty$.

Remark 2.2 These results can be generalized if in the equation (1) for $\alpha_{i-1}(t) \leq s \leq \alpha_i(t)$ $K(t, s) = K_i(t, s)$ where $\alpha_i(0) = 0$, $0 \leq \alpha'_0(0) < \alpha'_1(0) < \dots < \alpha'_n(0) \leq 1$, $i = 1, \dots, n$.

Conclusion

Our method allow studies of the Volterra integral equations of the first kind when kernals are discontinuous operator-functions operating in Banach spaces. Our work naturally complements the theory of integral models of evolving systems.

Acknowledgement

This work is fulfilled as part of Federal Framework Programm “Cadri” P696 (30.09.2010), it is also partly supported by RFBR, projects 09–01–00377 and 11-08-00109.

References

- [Apa03] A.S. Apartsyn. Nonclassical Linear Volterra Equations of the First Kind. (first edition in 2003 by the Walter de Gruyter), 2003.
- [DL95] A.M. Denisov and A. Lorenzi. On a special volterra integral equation of the first kind. Boll. Un. Mat. Ital. B., 7(9):443–457, 1995.
- [Eva10] G.C. Evans. Integral equation of the second kind with discontinuous kernel. Transactions of the American Mathematical Society, 11(4):393–413, 1910.
- [Gel59] A.O. Gelfond. Calculus of Finite Differences. Fizmatlit, 1959.

- [HY96] N. Hritonenko and Yu. Yanenko. Modeling and Optimization of the Lifetime of Technologies. *Kluwer Academic Publishers*, 1996.
- [HY03] N. Hritonenko and Yu. Yatsenko. Applied mathematical modelling of engineering problems. *Kluwer*, 2003.
- [Khr06] A.P. Khromov. *Integral operators with discontinuous kernel on piecewise linear curves*. Sbornik: Mathematics, 197(11):115–142, 2006.
- [Mag83] N.A. Magnitsky. *Asymptotics of the solution of the volterra integral equations of the first kind*. DAN USSR, 169(1):29–32, 1983.
- [MST11] E.V. Markova, I.V. Sidler, and V.V. Trufanov. *On models of developing systems and their applications*. Automation and Remote Control, 72(7):1371–1379, 2011.
- [SS06] N.A. Sidorov and D.N. Sidorov. *Generalized solutions to integral equations in the problem of identification of nonlinear dynamic models*. Differential Equations, 42(9):1312–1316, 2006.
- [SS11] N.A. Sidorov and D.N. Sidorov. *Small solutions of nonlinear differential equations near branching points*. Russian Mathematics, 55(5):43–50, 2011.
- [SST07] N.A. Sidorov, D.N. Sidorov, and A.V. Trufanov. *Existence and structure of solution of integral-functional volterra equations of the first kind*. Bulletin of Irkutsk State University: Mathematics, 1(1):267–274, 2007.
- [ST09] N.A. Sidorov and A.V. Trufanov. *Nonlinear operator equations with a functional perturbation of the argument of neutral type*. Differential Equations, 45(12):1840–1844, 2009.
- [Tre07] V.A. Trenogin. Functional Analysis. *Nauka, Moscow*, 2007.